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ASYMPTOTIC SOLUTION OF CERTAIN NONLINEAR PROBLEMS OF TIME-OPTIMAL RESPONSE

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The problem of optimal control of nonlinear oscillatory motions with unspecified instant of the control process termination is investigated. The motion is assumed to be specified by the conventional system with a rotating phase and frequency strongly dependent on the slow vector [1, 2]. It is assumed that the problem satisfies the principle of maximum [3]. The method of averaging [2] is used for formulating a two-point first approximation problem which provides the optimal solution and is simpler and of lower dimension. The proposed method can be used for solving applied control problems of the type of time-optimal response for nonlinear oscillatory systems by applying small but protracted control actions. Specific examples of mechanical nature are calculated.

Approximate analytical methods of solving optimal control problems, including asymptotic methods based on the concept of the small parameter and separation of motions were developed in [2, 4-7] and others. The case of the asymptotically slowly reached fixed instant of the control process termination was considered in [2], and in [6] the author investigated the quasi-linear problem similar to the one considered here.

1. Statement of the problem. We consider the problem of control by a system of conventional form [2, 6]

$$a^{\cdot} = \varepsilon f(a, \psi, u, \varepsilon), \quad a(t_0) = a_0 \tag{1.1}$$
$$\psi^{\cdot} = \omega(a) + \varepsilon F(a, \psi, u, \varepsilon), \quad \psi(t_0) = \psi_0$$

where a is the vector of slow variables of arbitrary dimension $n \geqslant 2$ in some bounded region; ψ is the fast scalar variable (the rotating phase), $|\psi| < \infty$; $\omega(a) \geqslant \omega_0 > 0$ is the frequency throughout the considered region of variation of a; $u \in U$ is the vector of controlling functions of dimension m and U is a fixed convex region; $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$ is a small numerical parameter; a dot denotes differentation with respect to time $t \in [t_0, t_1]$, $t_1 \sim \varepsilon^{-1}$; t_0, a_0, and ψ_0 are initial data. In the indicated region the right-hand sides of equations of system (1.1) are assumed determined and fairly smooth so that the substitution of admissible bounded piecewise continuous control functions u ensures the existence and uniqueness of solution of the system in the interval $t \in [t_0, t_1]$ for all $\varepsilon > 0$. The requirements for smoothness that are necessary for substantiation and application of the proposed method are defined more exactly later.

We pose the following problem of optimal control: the phase point of system (1.1) is to be transferred onto the manifold

$$M(a, \varepsilon)|_{t_1} = 0, \quad M = (M_1, \ldots, M_r), \quad 1 \leqslant r \leqslant n-1 \tag{1.2}$$

at some unspecified instant of time t_1 so that

$$J = g(a, \varepsilon)|_{t_1} \to \min \quad u \in U \tag{1.3}$$

where functions M and g are considered to be fairly smooth.

The essential feature of the above formulation of the problem is the assumption of the independence of functions M and g of the fast variable ψ. In the case of strong dependence of ω on a $(\omega'(a) \sim 1)$ in the averaging schemes [1, 2] the value of that variable is determined with reduced absolute accuracy which, generally speaking, does not allow in the opposite case to construct the first approximation boundary value problem. Moreover, on such assumption we have $t_1 \sim \varepsilon^{-1}$ which makes possible the use of the asymptotic method of averaging for an approximate solution of the boundary value problem of the maximum principle. It should be noted that in practical problems involving small but protracted control actions, the quantity ψ, i.e. the position of the object in the phase space is usually not specified. In such problems the interest is focused on the variation of motion parameters (the slow variables a) which remain when $\varepsilon = 0$. For example, in Kepler's limited plane problem [8, 9] such parameters are the energy and the moment of momentum or the orbit eccentricity and the focal parameter, and the initial true anomaly (see Example 2 in Sect. 3); in the problem of optimal control of an asymmetric solid body rotation about its center of mass (Euler's case) these are the moment of momentum and energy or other integrals of unperturbed motion [10] (see Example 3 in Sect. 3).

The approximate solution of the problem of optimal control (1.1) - (1.3) is constructed on the basis of the necessary conditions of the maximum principle [3] on the assumption that such solution exists and is unique for any $\varepsilon \in [0, \varepsilon_0]$. Then the condition of maximum of the problem Hamiltonian

$$H(a, \psi, p, q, u, \varepsilon) = \varepsilon (pf) + q(\omega + \varepsilon F) \equiv \omega q + \varepsilon h \tag{1.4}$$

with respect to $u \in U$ and fixed other arguments uniquely determine the unique optimal control

$$u = u^*(a, \psi, p, q, \varepsilon) \tag{1.5}$$

Function u^* is assumed to be fairly smooth and 2π-periodic with respect to ψ [2]. In (1.4) p and q are variables conjugate of a and ψ that satisfy the system of equations and the transversality conditions at the right-hand end

$$p^\cdot = -\frac{\partial H}{\partial a}\Big|_{u*} = -\omega' q - \varepsilon \frac{\partial h}{\partial a}\Big|_{u*}, \quad p(t_1) = -\frac{\partial}{\partial a}(\alpha M)_{t_1} \tag{1.6}$$

$$q^\cdot = -\frac{\partial H}{\partial \psi}\Big|_{u*} = -\varepsilon \frac{\partial h}{\partial \psi}\Big|_{u*}, \quad q(t_1) = 0$$

where α is a constant vector which is to be determined, and expressions of the kind (αM) and similar in (1.4) denote scalar products. Variables a and ψ are determined by the joint solution of the two-point problem (1.1), (1.2), (1.5) and (1.6).

The system of boundary conditions is closed by the relationship

$$H^* |_{t_1} = \varepsilon h\ (a, \psi, p, 0, u^*, \varepsilon) |_{t_1} = 0 \qquad (1.7)$$

which can be considered as the equation in the unknown t_1. Since the canonical system (1.1), (1.6) is automonmous, it follows from (1.7) that along the considered trajectories for all $t \in [t_0, t_1]$,

$$H^* = \omega q + \varepsilon h^* = 0 \qquad (1.8)$$

The asterisk indicates that the expression for u^* in (1.5) is substituted for u. The nonstandard system of equations (which contains in the first vector equation (1.6) the term $\omega' q$) is reduced to the standard form, as in [2], using the identity (1.8). We thus have to derive a solution of the boundary value problem which yields the minimum of functional (1.3).

2. Derivation of first approximation solution. Below we consider the solution of the boundary value problem which satisfies condition (1.7), and makes possible the reduction of Eqs. (1.1) and (1.6) to a system of the standard form that does not contain the variable q

$$a^{\cdot} = \varepsilon f_0^*\ (a, \psi, p) + \varepsilon^2 f_1, \quad \psi^{\cdot} = \omega\ (a) + \varepsilon F \qquad (2.1)$$

$$p^{\cdot} = -\varepsilon\omega'\ (a)\omega^{-1}\ (pf_0^*) - \varepsilon\partial\ (pf_0^*)\ /\ \partial a + \varepsilon^2 P_1$$

where $q = -\varepsilon\omega^{-1}\ (pf_0^*) + \varepsilon^2 Q_1$ is obtained by solving Eqs (1.8) for that variable, which is possible when ε is fairly small, since $\omega\ (a) \geqslant \omega_0 > 0$. In the considered region functions f_1, P_1 and Q_1 are uniformly bounded. Their explicit form is not adduced since it is unimportant in the case of first order solution, and terms $O\ (\varepsilon^2)$ in (2.1) can be disregarded [1, 2].

The algorithm of solution of the input boundary value problem consists of deriving a set of solutions of the shortened boundary value problem for system (2.1) with appertaining initial and boundary conditions (1.1), (1.2) and (1.6). The quantity t_1 can be conveniently taken as the parameter of the set. It is assumed below that the solution $a = a\ (t, t_1, \varepsilon)$, $p = p\ (t, t_1, \varepsilon)$, and $\psi = \psi\ (t, t_1, \varepsilon)$ of the boundary value problem exists and is unique for any specified $t_1\ (t_1 \sim \varepsilon^{-1})$. For brevity the dependence on a_0 and ψ_0 is not given. The complete solution of the input boundary value problem (see Sect. 1) is obtained when parameter t_1 is determined and satisfies the equation

$$h^* |_{t_1} \equiv (pf^*)_{t_1} = 0 \qquad (2.2)$$

If the admissible root $t_1\ (\varepsilon)\ (t_1 \sim \varepsilon^{-1})$ is unique, the obtained solution of the boundary value problem is also the solution of the problem of optimal control (1.1) - (1.3) [3]. However, as implied by (2.2), that equations has for small ε many roots whose number is generally of order $[\varepsilon^{-1}]$. This is due to that the left-hand side of Eq. (2.2) $h^*\ [t_1, \varepsilon]$ is an oscillating function of parameter t_1 whose frequency and amplitude are of the order of unity, and the mean value $\langle h^* \rangle$ over ψ is a slow varying function of t_1, i.e. $d\ \langle h^* \rangle\ /\ dt_1 \sim \varepsilon$. The typical behavior of h^* and $\langle h^* \rangle$ as functions of τ_1 ($\tau = \varepsilon t$ is the slow time) is shown in Fig. 1.

Among the discrete set of roots it is necessary to select that which minimizes the functional (1.3), i.e. $J^* = \min J$, $\tau_1^* \in \{\tau_1\}$. Subsequent analysis will show that the typical dependence of the functional on τ_1 is of the form shown in Fig. 2

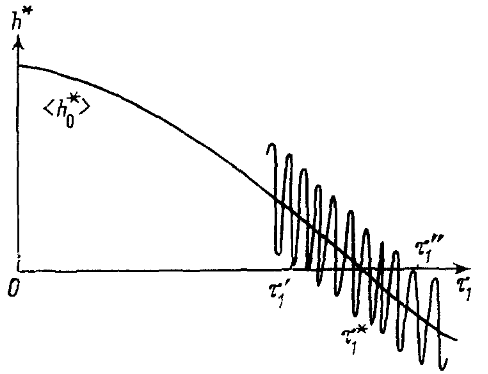

Fig. 1

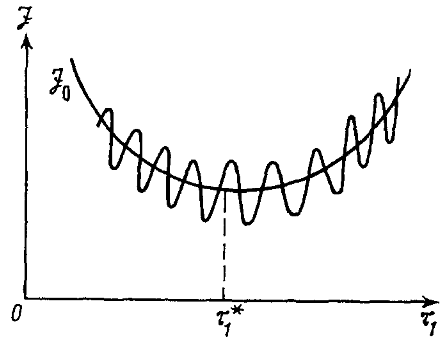

Fig. 2

(see Example 1 in Sect. 3). The asymptotically considerable number of solutions of the boundary value problem makes it difficult to use known numerical methods of solving problems of optimal control. The procedure described below makes possible the derivation of the approximate solution using slow variables and the functional with an accuracy $\sim\varepsilon$.

The corresponding to (2.1) averaged system of first approximation equations is obtained by rejecting terms of order ε^2 and averaging over the phase ψ with fixed slow variables a and p

$$d\xi \,/\, d\tau = f_0(\xi, \eta), \quad \tau \in [0, \tau_1], \quad \tau_1 \sim 1 \tag{2.3}$$
$$\frac{d\eta}{d\tau} = -\frac{\omega'(\xi)}{\omega(\xi)}(\eta f_0) - \left(\eta \frac{\partial f_0}{\partial \xi}\right)$$

where ξ and η are the averaged slow variables and f_0 is the mean value of function f_0^* over ψ. The system of initial and boundary conditions is of the form

$$\xi(0) = a_0, \quad M_0(\xi)_{\tau_1} = 0, \quad \eta(\tau_1) = -(\alpha M_0')_{\tau_1} \tag{2.4}$$
$$(M_0(\xi) \equiv M(\xi, 0))$$

Let us assume that the boundary value problem (2.3), (2.4) has the unique solution $\xi = \xi(\tau, \tau_1)$, $\eta = \eta(\tau, \tau_1)$ for any specified $\tau_1 \sim 1$, and that to a small variation of initial and boundary conditions of order $\sim\varepsilon$ in the right-hand sides of Eqs. (2.3) corresponds to a similar variation of solution. The difference between the solutions of the input boundary value problem and the problem averaged over slow variables is then of order ε [2]. Using successive approximations with respect to powers of parameter ε , it is possible to show that on condition of uniqueness of solution of the boundary value problems (2.1) and (2.3) indicated above and the fulfilment of smoothness requirements which ensure the ε -closeness of the corresponding Cauchy problems for the considered equations [1], a similar closeness exists for solutions of boundary value problems in any interval $t \in [t_0, t_1]$, $t_1 = L\varepsilon^{-1}$, where L is a constant which can be as large as desired and fairly small ε_0

$$a(t, t_1, \varepsilon) = \xi(\tau, \tau_1) + O(\varepsilon), \quad p(t, t_1, \varepsilon) = \eta(\tau, \tau_1) + O(\varepsilon) \tag{2.5}$$

$$q = \varepsilon\omega^{-1}(\xi)(\eta f_0^*(\xi, \psi, \eta)) + O(\varepsilon^2), \quad \psi = \frac{1}{\varepsilon}\int_0^\tau \omega(\xi(\tau', \tau_1))\, d\tau' + O(1)$$

The problem of optimal control is solved in the first approximation when the optimal instant τ_1 of the control process termination is defined with an error $O(\varepsilon)$. The equation

$$h^*\left(\xi(\tau_1, \tau_1) + O(\varepsilon), \frac{1}{\varepsilon}\int_0^{\tau_1} \omega(\xi(\tau, \tau_1))d\tau + O(1), \right. \tag{2.6}$$
$$\left. \eta(\tau_1, \tau_1) + O(\varepsilon), O(\varepsilon), \varepsilon\right) = 0$$

obtained from (2.2) by substituting into it expressions (2.5) is inadequate for determining τ_1 with the required accuracy, since the phase ψ is determined with an error $O(1)$. However, owing to the fast oscillation relative to τ_1 (with frequency $\sim\varepsilon^{-1}$ and amplitude of order unity) of the left-hand side of (2.6), established previously, it is possible to maintain that in the ε-neighborhood of each of roots $\{\tau_1\}$ calculated without regard to errors, there exists a root of Eq. (2.2). For this it is sufficient that the equation

$$\langle h_0(\xi(\tau_1, \tau_1), \quad \eta(\tau_1, \tau_1))\rangle = 0 \tag{2.7}$$

has a real root τ_1^*, and

$$\omega(\xi(\tau_1^*, \tau_1^*)) + \int_0^{\tau_1^*} \omega'(\xi(\tau, \tau_1^*)) \frac{\partial\xi(\tau, \tau_1^*)}{\partial\tau_1^*} d\tau \neq 0 \tag{2.8}$$

In what follows these conditions are assumed to be satisfied. Since it is sufficient to determine roots $\{\tau_1\}$ with an error $O(\varepsilon)$, the admissible values of τ_1 can be selected in the considered first approximation from a certain continuous interval $\tau_1 \in [\tau_1', \tau_1'']$ (see Fig. 1).

The optimal value of τ_1 is obtained from the condition (1.3) of minimum of functional J calculated with an error $O(\varepsilon)$ and considered to be a function of the continuous argument $\tau_1 \in [\tau_1', \tau_1'']$

$$J_0 = g_0(\xi(\tau_1, \tau_1)) \to \min_{\tau_1} \tag{2.9}$$

Below we present the necessary and sufficient conditions of optimum of τ_1 and solution (2.5) of system (2.3). First, it should be noted that the system admits the integral

$$\varkappa = -(\eta f_0(\xi, \eta))\omega^{-1}(\xi) \tag{2.10}$$

which shows that the mean value of q in the interval $\Delta t \sim \varepsilon^{-1}$ is constant within an error $O(\varepsilon^2)$. This is so, since differentiation of (2.10) with respect to τ yields the identity $d\varkappa / d\tau \equiv 0$ by virtue of Eqs. (2.3) with allowance for the relationship $\partial(\eta f_0) / \partial\eta = f_0$. As the result, system (2.3) can be represented in the equivalent form

$$\frac{d\xi}{d\tau} = f_0(\xi, \eta), \quad \frac{d\eta}{d\tau} = -\omega'(\xi)\varkappa - \frac{\partial}{\partial\xi}(\eta f_0) \tag{2.11}$$

where $\varkappa$ is the unknown parameter related to the sought quantity τ_1 by the condition [6]

$$[\omega(\xi)\varkappa + (\eta f_0(\xi, \eta))]_{\tau_1} = 0 \tag{2.12}$$

where $\xi = \xi(\tau, \tau_1, \varkappa)$, $\eta = \eta(\tau, \tau_1, \varkappa)$ is the solution of the boundary value problem for Eqs. (2.11). If (2.12) is solved for $\tau_1 = \tau_1(\varkappa)$, then, after the substitution into condition (2.6), we obtain that in the ε-neighborhood of any $\varkappa$ that belongs to some neighborhood of point $\varkappa = 0$: $\varkappa \in [\varkappa', \varkappa'']$, $\varkappa' < 0$, $\varkappa'' > 0$ under condition

$$\frac{d}{d\varkappa}\int_0^{\tau_1(\varkappa)} \omega(\xi(\tau, \tau_1(\varkappa), \varkappa))\,d\tau \neq 0 \tag{2.13}$$

there exists a root of the exact equation (2.2).

Thus the minimization of functional J_0 (2.9) with respect to $\tau_1 \in [\tau_1', \tau_1'']$ reduces to the minimization with respect to $\varkappa \in [\varkappa', \varkappa'']$ with allowance for condition (2.12). The necessary condition of a local minimum is that the total derivative with respect to $\varkappa$

$$dJ_0 / d\varkappa = -(\eta_1 d\xi_1 / d\varkappa), \quad \xi_1 = \xi_{\tau_1}, \quad \eta_1 = \eta_{\tau_1}$$

be zero. Use is made here of the identity $M_0(\xi_1) \equiv 0$, i.e. $dM_0 / d\varkappa|_{\tau_1} = 0$. The derivative $d\xi_1 / d\varkappa$ is

$$\frac{d\xi_1}{d\varkappa} = \frac{d\xi}{d\tau}\bigg|_{\tau_1}\frac{d\tau_1}{d\varkappa} + \frac{D\xi}{D\varkappa}\bigg|_{\tau_1}, \quad \frac{D\xi}{D\varkappa} = \frac{\partial\xi}{\partial\tau_1}\frac{d\tau_1}{d\varkappa} + \frac{\partial\xi}{\partial\varkappa}$$

It follows from (2.11) and (2.12) that

$$\frac{dJ_0}{d\varkappa} = \varkappa\frac{d}{d\varkappa}\int_0^{\tau_1} \omega(\xi(\tau, \tau_1, \varkappa))\,d\tau, \quad \tau_1 = \tau_1(\varkappa) \tag{2.14}$$

Formula (2.14) implies that $\varkappa = 0$ is a suspected extremal point. Furthermore, if

$$\frac{d^2J_0}{d\varkappa^2}\bigg|_{\varkappa=0} = \frac{d}{dx}\int_0^{\tau_1} \omega(\xi(\tau, \tau_1, \varkappa))\,d\tau\bigg|_{\varkappa=0,\ \tau_1=\tau_1(0)} > 0 \tag{2.15}$$

(see (2.8) and (2.13)), then $\varkappa = 0$ is a point of local minimum. In this formula $\tau_1(0)$ is the positive root of Eq. (2.12) determined for $\varkappa = 0$. If in the whole neighborhood of $\varkappa \in [\varkappa', \varkappa'']$

$$\varkappa\int_0^{\tau_1} \omega(\xi)\,d\tau \geqslant \int_0^{\varkappa} d\varkappa_1 \int_0^{\tau_1'(\varkappa_1)} \omega(\xi(\tau, \tau_1(\varkappa_1), \varkappa_1))\,d\tau \tag{2.16}$$

then $\varkappa = 0$ is the point of absolute minimum of functional $J_0(\varkappa)$, since

$$J_0(\varkappa) = J_0(0) + \varkappa\int_0^{\tau_1} \omega(\xi)\,d\tau - \int_0^{\varkappa} d\varkappa_1 \int_0^{\tau_1'(\varkappa_1)} \omega(\xi(\tau, \tau_1(\varkappa_1), \varkappa_1))\,d\tau$$

The establishment of formula (2.14) is the basic result of the analysis of the problem of optimal control (1.1) - (1.3).

Note that because $\omega(\xi)\varkappa + (\eta f_0) = \text{const}$ the system (2.11) is Hamiltonian, and since $\varkappa = 0$ is the optimal value, it follows from the condition (2.12) of transversality that $(\eta f_0(\xi, \eta)) = 0$. Here ξ, η is the solution of system (2.11) for $\varkappa = 0$, and $\tau_1 = \tau_1(0)$ is the root of the equation $(\eta(\tau_1, \tau_1, 0) \cdot f_0(\xi(\tau_1, \tau_1, 0), \eta(\tau_1, \tau_1, 0))) = 0$. Thus if the solution of Eq. (2.10) with $\varkappa = 0$ for some variable is substituted into system (2.11) (with $\varkappa = 0$), the order of the latter is reduced by one. The solution of the problem of optimal control of a system with one degree of freedom reduces to a quadrature and the final equation that determines $\tau_1(0)$. The above exposition implies that the order of the integrable input system of a $(2n + 2)$-nd order can be reduced by three. What is particularly important is that the slow variables are integrated independently of the fast phase. This makes possible the introduction of the slow time $\tau = \varepsilon t$ with the resulting considerable reduction of the volume of calculations required for numerical solution of the boundary value problem.

The analysis of the optimal control problem in Sect. 1 and 2 yields the approximate solution

$$u = u_0^*(a, \psi, \eta^*), \quad J^* = J_0(0) \tag{2.17}$$
$$\xi^* = \xi(\tau, \tau_1(0), 0), \quad \eta^* = \eta(\tau, \tau_1(0), 0)$$

Investigation of specific problems of motion control of nonlinear oscillating systems shows that the proposed method is applicable even in certain cases in which function u^* (1.5) is discontinuous with the number of discontinuities of the first kind $\sim [\varepsilon^{-1}]$ in the considered time interval [3, 11]. In such problems the effect of the quantity $q \sim \varepsilon$ has little effect $(\sim\varepsilon)$ on the control and the functional: its optimal minimum average value is directly obtained from the equations of motion. As the result of application of the averaging method it is possible to prove in first approximation the occurrence of local optimal controls; for solving the problem it is necessary to integrate Eqs. (1.1) with known discontinuous right-hand sides. Substantiation of the use of the averaging method in the case of such systems may be derived on the basis of [12], where the convential system [1] was considered.

A one-to-one correspondence is established between ψ and t when $\omega(a) \geqslant \omega_0 > 0$ and ε is fairly small, hence by dividing the system by $\psi^\cdot = \omega(a) + O(\varepsilon)$ it reduces to the conventional form $x^\cdot = \varepsilon X(t, x) + O(\varepsilon^2)$, for which the statement in [12] holds.

Let function $X(t, x)$ be uniformly bounded, and

$$\int_{t_0}^{t} dt' \int_{c}^{x} X(t', x')\, dx', \quad t \in [t_0, \infty), \quad x, c \in D$$

be continuously differentiable; there exist the uniform limits

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} X(t, x)\, dt = X_0(X), \quad x \in D$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} [Z(t, x_1, \ldots, x_i + r_k, \ldots, x_n) - Z(t, x)]\, r_k^{-1} dt = 0,$$

$$Z(t, x) \equiv X(t, x) - X_0(x)$$

where $x + r_k \in D$ and r_k is a decreasing sequence of numbers. It is assumed that $X_0(x)$ satisfies the Lipschitz condition for $x \in D$, that the solution $\xi(\tau)$ of equation $d\xi / d\tau = X_0(\xi)$ lies in D for $\tau \in [0, \infty)$ together with its ρ-neighborhood, and that the solution $x(t, \varepsilon)$ of the input system that satisfies condition $x(t_0, \varepsilon) = \xi(0)$ is unique. Then for any $\eta > 0$ and L as large as desired there exists such $\varepsilon_0 > 0$ that when $0 < \varepsilon < \varepsilon_0$ for the solution $x(t, \varepsilon)$ we have the inequality $|x(t, \varepsilon) - \xi(\tau)| < \eta$ for $t \in [t_0, L\varepsilon^{-1}]$.

This theorem and the similar one for the case of smooth right-hand sides [1] does not ensure the estimate $|x - \xi| \sim \varepsilon$. If the conditions formulated above are supplemented by the requirement for function $\int Z(t, x)\, dt$ to have continuous derivatives with respect to $x \in D$, function N, which is the mean value of function Z with respect to t in the interval $[t_0, t_0 + T]$, decreases fairly rapidly as $T \to \infty$, i.e. there exist such constants $T_0 > 0$ and $d_0 > 0$ that the inequality $|N| \leqslant d_0 T^{-1}$ is uniformly satisfied throughout the considered region for $T > T_0$. The more accurate estimate $|x - \xi| \leqslant d\varepsilon$ is then valid when $t \in [t_0, L\varepsilon^{-1}]$, where $0 < d < \infty$.

Note that the general scheme of obtaining higher approximations with respect to powers of ε [1], in particular the canonical method of averaging [6], can be applied to system (1.1), (1.6).

3. Examples. 1. Let us consider a rotation system with one degree of freedom that is close to the conservative system

$$m(x)\, x^{\cdot\cdot} + {}^1/_2 m' x^{\cdot 2} + V'(x) = \varepsilon u + \varepsilon f(x, x^\cdot) \qquad (3.1)$$

$$\infty > x_2^\cdot > x^\cdot \geqslant x_1^\cdot > 0, \quad \infty > m_2 \geqslant m \geqslant m_1 > 0$$

$$\infty > V_2 \geqslant V \geqslant V_1 > -\infty$$

where x is the generalized coordinate, $x^\cdot = dx / dt$ is the speed of rotation, $m(x)$ is the "mass", $V(x)$ is the perturbing function of the 2π-periodic function f of x, and u is the scalar control. It is assumed that m, V and f are fairly smooth functions. The initial conditions are: $x(0) = x_0$ and $x^\cdot(0) = x_0^\cdot > 0$.

Equation (3.1) is integrable when $\varepsilon = 0$

$$a = {}^1/_2 m x^{\cdot 2} + V, \quad t + \gamma = \int dx / x^\cdot(a, x)$$

where a is the total energy and γ is the phase constant. Using the substitution $(x, x^\cdot) \to (a, \psi)$ we reduce system (3.1) to the form

$$a^\cdot = \varepsilon x^\cdot (u + f), \quad x^\cdot(a, x) = (2 / m)^{1/2} (a - V)^{1/2}, \quad a(0) = a_0$$

$$\psi^\cdot = \omega(a) + O(\varepsilon), \quad \omega(a) = 2\pi / \oint dx / x^\cdot, \quad \psi(0) = \psi_0$$

Since the general unperturbed solution is known only implicitly, the first approximation averaging can be effected using the relationship $d\psi = \omega(a)\, dx / x^{\cdot}$.

When the considered problem is $|u| \leqslant u_0$, $M = a - a_1$ $(a_1 > V_2)$, $g = \varepsilon t$ and the mean of external forces εf per revolution is zero, $\langle f(\xi)\rangle \equiv 0$, the approximate solution

$$u^* = u_0 \operatorname{sign}(a_1 - a_0), \quad u_0 \tau \operatorname{sign}(a_1 - a_0) = \int_{a_0}^{\xi} \frac{d\xi'}{\omega(\xi')}$$

$$\omega(\xi) = \langle x^{\cdot}(\xi)\rangle, \quad \tau_1 = \tau(a_1, a_0), \quad \tau = \varepsilon t$$

is unique, and when $V = \text{const}$

$$\xi^{1/2} = a_0^{1/2} + (a_1^{1/2} - a_0^{1/2})\,\tau\tau_1^{-1}, \quad \tau_1 = |a_1^{1/2} - a_0^{1/2}|\, u_0^{*-1}$$

If, however, $\langle f(\xi)\rangle \not\equiv 0$, the solution is determined by the equation for ξ with the initial and boundary conditions

$$d\xi / d\tau = u_0 \omega(\xi) \operatorname{sign}(a_1 - a_0) + \langle f(\xi)\rangle, \quad \xi(0) = a_0, \quad \xi(\tau_1) = a_1$$

That problem not always has a solution, i.e. $\tau_1 > 0$.

To illustrate the method developed in Sect. 2 we present the analysis of a "smooth" system in which $m(x) \equiv 1$ and $V = \text{const}$. The terminal condition (1.2) is to have $x^{\cdot}(\tau_1) = v_1$ $(x^{\cdot}, v_1 > 0)$, and the functional is of the form $J = b(\tau_1)$ where $db / d\tau = k + u^2$, $b(0) = 0$, and $k > 0$. In such formulation x is the phase and $x^{\cdot}$ is the slow variable. In accordance with Sect. 1 $u^* = {}^1/_2 p$, where p is the conjugate variable of $x^{\cdot}$. Solution of the boundary value problem of the kind (2.11) is

$$\xi(\tau, \tau_1, \varkappa) = (\varkappa / 4)(\tau - \tau_1)^2 - (\tau - \tau_1)\,\tau_1^{-1}(\Delta v + \varkappa\tau_1^2 / 4) + v_1 \qquad (3.2)$$

$$\eta(\tau, \tau_1, \varkappa) = -\varkappa(\tau - \tau_1) - 2\tau_1^{-1}(\Delta v + \varkappa\tau_1^2 / 4), \quad \Delta v \equiv v_0 - v_1$$

On the basis of (3.2)

$$J_0(\tau_1, \varkappa) = 2k\tau_1 - \varkappa^2\tau_1^3 / 24 - \varkappa\tau_1\Delta v / 2 - \varkappa v_1\tau_1$$

We have to determine the minimum of J_0 with respect to $\varkappa$ and with allowance for condition of the kind (2.12), $(v_1\varkappa - k)\,\tau_1^2 + (\Delta v + \varkappa\tau_1^2 / 4)^2 = 0$. As expected the implicit derivative of J_0 at point $\varkappa = 0$ is zero, since

$$\left.\frac{dJ_0}{d\varkappa}\right|_{\varkappa=0} = 2k \left.\frac{d\tau_1}{d\varkappa}\right|_{\varkappa=0} - \tau_1(0)\frac{\Delta v}{2} - v_1\tau_1(0),$$

$$\left.\frac{d\tau_1}{d\varkappa}\right|_{\varkappa=0} = \frac{\tau_1(0)}{2k}\left(v_1 + \frac{\Delta v}{2}\right)$$

The second derivative (see (2.15)) is positive

$$\left.\frac{d^2J_0}{d\varkappa^2}\right|_{\varkappa=0} = \frac{\tau_1^3(0)}{24} + \frac{\tau_1(0)}{2k}\left(v_1 + \frac{\Delta v}{2}\right)^2 > 0$$

The approximate solution of the optimum control problem is of the simple form

$$u^* = -\Delta v\tau_1^{-1}(0), \; J_0^* = 2k^{1/2}|\Delta v|, \; \tau_1(0) = |\Delta v|\, k^{-1/2}, \; \xi = v_0 - \Delta v\tau_1^{-1}(0)\,\tau$$

2. We consider the controllable plane motion of a point in a gravitational field [8,10]. In dimensionless variables of time t and polar coordinates (r, φ) the

equation of motion is of the form

$$r^{\cdot} = v_r, \quad v_r^{\cdot} = v_\varphi^2 r^{-1} - r^{-2} + \varepsilon u_r, \quad r(0) = r_0, \quad v_r(0) = v_{r0} \tag{3.3}$$
$$\varphi^{\cdot} = v_\varphi r^{-1}, \quad v_\varphi^{\cdot} = -v_r v_\varphi r^{-1} + \varepsilon u_\varphi, \quad \varphi(0) = \varphi_0, \quad v_\varphi(0) = v_{\varphi 0}$$

where u_r and u_φ are control functions and ε is a small parameter. When $\varepsilon = 0$ the system is integrable

$$\tfrac{1}{2}(v_r^2 + v_\varphi^2) - r^{-1} = E, \; rv_\varphi = K, \; r = p\,(1 + e\cos x)^{-1} \tag{3.4}$$
$$r = \tfrac{1}{2}\,|E|^{-1}(1 - e\cos\xi), \; t + \delta = (2\,|E|)^{-3/2}(\xi - e\sin\xi)$$

where $E < 0$ is the energy, $K > 0$ is the moment of momentum, $p = K^2$ is the focal parameter, $0 < e = (1 + 2EK^2)^{1/2} < 1$ is the orbit eccentricity, $x = \varphi - \gamma$, and γ and δ are arbitrary constants. Motion of the point is bounded and periodic of period T that depends only on energy $T(E) = 2\pi\,(2\,|E|)^{-3/2}$. The angular variable φ during a period obtains a 2π increment since

$$x = \psi - \int_0^x \left[\frac{(1 - e^2)^{3/2}}{(1 + e\cos y)^2} - 1\right] dy, \quad \psi = \frac{2\pi}{T}(t + \delta) \tag{3.5}$$

Formulas (3.4) and (3.5) make it possible to pass to the following system of the kind (1.1) when $\varepsilon \neq 0$:

$$E^{\cdot} = \varepsilon(v_r u_r + v_\varphi u_\varphi), \quad K^{\cdot} = \varepsilon r u_\varphi \tag{3.6}$$
$$\gamma^{\cdot} = \varepsilon e^{-1} p^{1/2}[u_r \cos x + (2 + e\cos x)(1 + e\cos x)^{-1} u_\varphi \sin x]$$
$$\psi^{\cdot} = 2\pi T^{-1} - (1 - e^2)^{3/2}(1 + e\cos x)^{-2}\gamma^{\cdot} + \{(1 + e)(1 + e\cos x)^{-1}\sin x\,\partial\,[(1 - e)^{1/2}(1 + e)^{-1/2}]\,/\,\partial e - \partial\,[e\,(1 - e^2)^{1/2}(1 + e\cos x)^{-1}]/\partial e\,\sin x\}\,e^{\cdot}$$

Initial data for (3.6) are determined by formulas (3.3) - (3.5), and

$$e^{\cdot} = \varepsilon p^{1/2}\{u_r \sin x + u_\varphi\,[e\,(1 + \cos^2 x) + 2\cos x]\,(1 + e\cos x)^{-1}\}$$
$$p^{\cdot} = 2\varepsilon p^{3/2} u_\varphi (1 + e\cos x)^{-1}$$

For (3.6) we have the following optimal control problem:

$$E(t_1) = E_1 \leqslant 0, \quad J = l\tau_1 + \int_0^{\tau_1} u^2 d\tau \tag{3.7}$$
$$(l > 0, \quad u^2 = u_r^2 + u_\varphi^2)$$

Formulation of the boundary value problem is based on the considerations in Sect. 1. Since the angular variable does not in fact appear in the right-hand side of system, its conjugate variable p_γ is identically zero, since $p_\gamma^{\cdot} = -\partial H\,/\,\partial\gamma = 0$ and $p_\gamma(t_1) = 0$. Then from the estimate $p_\psi = O(\varepsilon)$ follows that for $t_1 \sim \varepsilon^{-1}$ in the first approximation

$$u_r^* = \tfrac{1}{2} p_E v_r, \quad u_\varphi^* = \tfrac{1}{2}(p_E v_\varphi + p_K r) \tag{3.8}$$

where p_ψ, p_E, and p_K are the respective conjugate variables. The averaging of system (3.6) is similar to that in Example 1 and is carried out using the relationship $d\psi = 2\pi T^{-1} p^{3/2}(1 + e\cos x)^{-2}\,dx$. The averaged system implies that $p_K \equiv 0$, since

$$\frac{dp_K}{d\tau} = -\left(\frac{p_K}{4}\frac{\partial\langle r^2\rangle}{\partial K} + p_E\frac{\partial\langle v_\varphi r\rangle}{\partial K}\right)p_K, \quad p_K(\tau_1) = 0$$

Here and subsequently the old notation is used for the averaged variables. As the result, we obtain the explicit solution of the boundary value problem and the formula for the controls (3.8)

$$E(\tau) = E_0\left(1 + \frac{\rho\tau}{2-\rho\tau_1}\right), \quad p_E(\tau) = \frac{-2\rho}{2-(\tau_1-\tau)\rho} \tag{3.9}$$
$$K(\tau) = K_0(1 - {}^1/_2\rho\tau_1)[1 - {}^1/_2\rho(\tau_1-\tau)]^{-1}, \quad \tau_1 = \sqrt{2}\,\|E_1|^{1/2} - |E_0|^{1/2}|\,l^{-1/2}$$
$$\rho = (2l/|E_1|)^{1/2}\operatorname{sign}[1-(E_0/E_1)^{1/2}], \quad u_r^* = {}^1/_2p_E(\tau)v_r, \quad u_\varphi^* = {}^1/_2p_E(\tau)v_\varphi$$

where velocities v_r and v_φ may be expressed in terms of φ and γ with $\gamma = \gamma_0 = \text{const}$, since it follows from (3.6) that the mean value of $\gamma^\cdot$ is zero. Hence

$$v_r = ep^{-1}K\sin(\varphi-\gamma_0), \quad v_\varphi = p^{-1}K[1 + e\cos(\varphi-\gamma_0)]$$

The averaged slow variables e and p are defined in (3.9) in terms of $E(\tau)$ and $K(\tau)$. If the eccentricity e is "small", the radial control component $u_r^* \sim e$ is also small, while the transversal component $u_\varphi^* \sim 1$ is a slow varying function on which are superimposed small vibrations of amplitude $\sim e$.

3. The proposed method can be applied for solving certain problems of optimal control of rotation of a solid body relative to its center of mass [10] (Euler's problem). The motion is considered in a system of coordinates attached to the principal central axes of inertia, and the control is effected by small moments εM_x, εM_y, and εM_z. If the body is symmetric about the z-axis, i.e. the moments of inertia satisfy the condition $I_x = I_y \neq I_z$, the equations of motion are of the form

$$\begin{aligned}
&\omega_x^\cdot + (d-1)\omega_y\omega_z = \varepsilon u_x, \quad u_x = M_xI_x^{-1}, \quad \omega_x(0) = \omega_{x0}\\
&\omega_y^\cdot + (1-d)\omega_x\omega_z = \varepsilon u_y, \quad u_y = M_yI_y^{-1}, \quad \omega_y(0) = \omega_{y0}\\
&\omega_z^\cdot = \varepsilon u_z, \quad d = I_zI_x^{-1} \neq 1, \quad u_z = M_zI_z^{-1}, \quad w_z(0) = \omega_{z0}
\end{aligned} \tag{3.10}$$

The substitution

$$\omega_x = a\cos\psi, \; \omega_y = a\sin\psi, \quad w_z = c \;\; (a > 0,\; c \neq 0)$$

reduces system (3.10) to the conventional form (1.1)

$$a^\cdot = \varepsilon(u_x\cos\psi + u_y\sin\psi), \quad c^\cdot = \varepsilon u_z, \quad a(0) = a_0 = (\omega_{x0}^2 + \omega_{y0}^2)^{1/2}$$
$$\psi^\cdot = (d-1)c + \varepsilon a^{-1}(u_y\cos\psi - u_x\sin\psi), \quad c(0) = c_0 = \omega_{z0}, \; \psi(0) = \psi_0$$

Let it be necessary to slow down to zero the speed of rotation about the x- and y-axes in the shortest possible time t_1 with arbitrary $\omega_z \neq 0$, and let the controls be bounded by the ellipsoid $u_x^2l_x^{-2} + u_y^2l_y^{-2} + u_z^2l_z^{-2} \leqslant 1$, where l_x, l_y, and l_z are constants. The approximate set of controls is then obtained in the explicit form

$$u_x = -l_x\omega_xr^{-1}, \quad u_y = -l_y\omega_yr^{-1}, \quad u_z \equiv 0; \quad r = (l_x^2\omega_x^2 + l_y^2\omega_y^2)^{1/2}$$

The solutions of equations and the minimal time are also obtained in explicit form

$$a = a_0 - \langle R\rangle\tau, \; c = c_0, \; \tau_1 = a_0\langle R\rangle^{-1}$$

$$\langle R\rangle = 2\frac{l_x}{\pi}\begin{cases}\mathrm{E}(\sqrt{k}), & 0 \leqslant k < 1, \quad k = 1 - l_y^2 l_x^{-2}\\ \sqrt{1-k}\,\mathrm{E}(\sqrt{k/(k-1)}), & 0 > k > -\infty\end{cases}$$

where E is a complete integral of the second kind. The same solution applies to this problem when the body is only close to a dynamically symmetric, since the mean of small additions to the gyroscopic terms is zero.

Let now the set of controls be bounded by a cylindrical region $u_x^2 + u_y^2 \leqslant l^2$ and $|u_z| \leqslant h$. Such control scheme consists of a pair of fixed motors developing a limited moment about the axis of symmetry z and of a pair of vernier motors on the z-axis which develop moments limited by a circle about the x- and y-axes. Let, furthermore, the boundary conditions be of the form $a(t_1) = a_1$, $c(t_1) = c_1 = \omega_{z1}$. The problem thus stated leads to the particular controls $u_x = \cos\psi \operatorname{sign} p$, $u_y = \sin\psi \operatorname{sign} p$, and $u_z = \operatorname{sign} q$, where $\operatorname{sign} p$ and $\operatorname{sign} q$ are piecewise constant functions of τ which have a finite number of discontinuities, and are such that

$$\int_0^{\tau_1} \operatorname{sign} p\, d\tau = \frac{a_1 - a_0}{l}, \quad \int_0^{\tau_1} \operatorname{sign} q\, d\tau = \frac{\omega_{z1} - \omega_{z0}}{h}$$

$$\tau_1 = \max\{|a_1 - a_0|\, l^{-1}, \quad |\omega_{z1} - \omega_{z0}|\, h^{-1}\}$$

Some other problems of control of solid body rotation using small control moments with allowance for perturbing forces of various nature (gyroscopic, gravitational, viscous friction, etc.) can be solved similarly.

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